# Exercise 26

Prove the statement using the $\varepsilon$, $\delta$ definition of a limit.

$$\lim_{x \to 0} x^3 = 0$$

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### Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x - 0| < \delta \qquad \text{then} \qquad |x^3 - 0| < \varepsilon$$

for all positive $\varepsilon$. Start by working backwards, looking for a number $\delta$ that's greater than $|x|$.

$$|x^3 - 0| < \varepsilon$$

$$|x^3| < \varepsilon$$

$$|x|^3 < \varepsilon$$

$$\sqrt[3]{|x|^3} < \sqrt[3]{\varepsilon}$$

$$|x| < \sqrt[3]{\varepsilon}$$

Choose $\delta = \sqrt[3]{\varepsilon}$. Now, assuming that $|x| < \delta$,

$$\begin{aligned} |x^3 - 0| &= |x^3| \\ &= |x|^3 \\ &= |x||x||x| \\ &< (\delta)(\delta)(\delta) \\ &= (\sqrt[3]{\varepsilon})(\sqrt[3]{\varepsilon})(\sqrt[3]{\varepsilon}) \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x \to 0} x^3 = 0.$$